

Delooping of relative exact categories

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Abstract

We introduce a delooping model of relative exact categories. It gives us a condition that the negative K -group of a relative exact category becomes trivial.

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Introduction

The negative K -theory $\mathbb{K}(\mathcal{E})$ for an exact category \mathcal{E} is introduced in [Sch04], [Sch06] and [Sch11] by M. Schlichting, and also for a differential graded category and for a stable infinity category is innovated by C. Cisikinsi and G. Tabuada, A.J. Blumberg and D. Gepner and G. Tabuada in [CT11] and [BGT13] respectively. This generalizes the definition of Bass, Karoubi, Pedersen-Weibel, Thomason, Carter and Yao. The first motivation of our work is to investigate some vanishing conjectures of such negative K -groups:

- (a) For any noetherian scheme X of Krull dimension d , $K_{-n}(X)$ is trivial for $n > d$ ([Wei80]).
- (b) The negative K -groups of a small abelian category is trivial ([Sch06]).
- (c) For a finitely presented group G , $K_{-n}(\mathbb{Z}G) = 0$ for $n > 1$ ([Hsi84]).

In [Sch06] Corollary 6, it was given a description of $\mathbb{K}_{-1}(\mathcal{E})$ and a condition on vanishing of $\mathbb{K}_{-1}(\mathcal{E})$ for an (essentially small) exact category \mathcal{E} in terms of its *unbounded* derived category $\mathcal{D}(\mathcal{E})$: We have $\mathbb{K}_{-1}(\mathcal{E}) = \mathbb{K}_0(\mathcal{D}(\mathcal{E}))$ and $\mathbb{K}_{-1}(\mathcal{E})$ is trivial if and only if $\mathcal{D}(\mathcal{E})$ is idempotent complete (= Karoubian in the sense of [TT90], A.6.1). To extend this observation, we shall introduce the notion of *higher derived categories* $\mathcal{D}_n(\mathcal{E})$ and show the following theorem:

Theorem 0.1 (A special case of Corollary 3.2). *For an exact category \mathcal{E} , we have $\mathbb{K}_{-n}(\mathcal{E}) = \mathbb{K}_0(\mathcal{D}_n(\mathcal{E}))$. Moreover, $\mathbb{K}_{-n}(\mathcal{E})$ is trivial if and only if $\mathcal{D}_n(\mathcal{E})$ is idempotent complete.*

Recall that the derived category $\mathcal{D}(\mathcal{E})$ is the triangulated category obtained by formally inverting quasi-isomorphisms in the category of chain complexes $\mathbf{Ch}(\mathcal{E})$. The pair $(\mathbf{Ch}(\mathcal{E}), \text{qis})$ of the category of chain complexes on \mathcal{E} and the class of all quasi-isomorphisms in $\mathbf{Ch}(\mathcal{E})$ forms a complicial Waldhausen exact category. More generally, for a pair $\mathbf{E} = (\mathcal{E}, w)$ of an exact category \mathcal{E} and a class of morphisms w in \mathcal{E} which is closed under finite compositions and satisfies the *strict axiom* (cf. 1.1), we define a class of weak equivalences qw in the category of chain complexes $\mathbf{Ch}(\mathcal{E})$, which is called *quasi-weak equivalences* associated with w . If w is the class of isomorphisms in \mathcal{E} , then qw is just the class of quasi-isomorphisms on $\mathbf{Ch}(\mathcal{E})$. The derived category $\mathcal{D}(\mathbf{E})$ of \mathbf{E} is obtained by formally inverting the quasi-weak equivalences in $\mathbf{Ch}(\mathcal{E})$. Put $\mathbf{Ch}(\mathbf{E}) = (\mathbf{Ch}(\mathcal{E}), qw)$ and one can define the class of weak equivalences in $\mathbf{Ch}_n(\mathbf{E}) := \mathbf{Ch}(\mathbf{Ch}_{n-1}(\mathbf{E}))$ inductively. The n -th derived category $\mathcal{D}_n(\mathbf{E})$ associated with \mathbf{E} , is the derived category of $\mathbf{Ch}_n(\mathbf{E})$. We also obtain the following theorems on the negative K -theory $\mathbb{K}(\mathbf{E})$ for a very strict consistent relative exact category \mathbf{E} (for definition, see 1.1 and 1.7):

Theorem 0.2 (A special case of Theorem 2.6). $\mathbb{K}(\mathbf{Ch}(\mathbf{E})) \xrightarrow{\sim} \Sigma \mathbb{K}(\mathbf{E})$, where Σ is a suspension functor on the stable category of spectra.

The organization of this note is as follows: In Section 1, we define the derived categories of \mathbf{E} and introduce the notion of quasi-weak equivalences. We will prove Theorems 0.2 and 0.1 in Sections 2 and 3 respectively.

Conventions. Throughout the paper, we use the letters \mathcal{E} and w to denote an essentially small exact category and a class of morphisms in \mathcal{E} . We write \mathbf{E} for the pair (\mathcal{E}, w) . For any category \mathcal{C} , we denote the class of all isomorphisms in \mathcal{C} by $i_{\mathcal{C}}$ or simply i . For any additive category \mathcal{B} , we denote the category of bounded (resp. unbounded above and bounded below) chain complexes on \mathcal{B} by $\mathbf{Ch}^b(\mathcal{B})$ (resp. $\mathbf{Ch}(\mathcal{B})$, $\mathbf{Ch}^+(\mathcal{B})$ and $\mathbf{Ch}^-(\mathcal{B})$). For any additive category \mathbf{B} we denote the idempotent completion of \mathbf{B} by \mathbf{B}^\sim . For any essentially small triangulated category \mathcal{T} , we denote the Grothendieck group of \mathcal{T} (resp. \mathcal{T}^\sim) by $K_0(\mathcal{T})$ (resp. $\mathbb{K}_0(\mathcal{T})$). We say that a sequence of triangulated categories $\mathcal{T} \xrightarrow{i} \mathcal{T}' \xrightarrow{j} \mathcal{T}''$ is *exact* if i is fully faithful, the composition ji is zero and the induced functor from $j, \mathcal{T}' / \mathcal{T} \rightarrow \mathcal{T}''$ is *cofinal*. The last condition means that it is fully faithful and every object of \mathcal{T}'' is a direct summand of an object of $\mathcal{T}' / \mathcal{T}$. For any complicial exact category with weak equivalence $\mathbf{C} = (\mathcal{C}, w)$, we write $\mathcal{T}\mathbf{C} = \mathcal{T}(\mathcal{C}, w)$ and $\mathfrak{F}\mathbf{C}$ for the associated triangulated category and the countable envelope of \mathbf{C} (See [Sch11, §3.2]).

1 Derived categories

1.1 (Relative exact categories). (1) A *relative exact category* $\mathbf{E} = (\mathcal{E}, w)$ is a pair of an exact category \mathcal{E} with a specific zero object 0 and a class of morphisms in \mathcal{E} which satisfies the following two axioms.

(Identity axiom). For any object x in \mathcal{E} , the identity morphism id_x is in w .

(Composition closed axiom). For any composable morphisms $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ in \mathcal{E} , if a and b are in w , then ba is also in w .

(2) A *relative exact functor* between relative exact categories $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow (\mathcal{F}, v)$ is an exact functor $f : \mathcal{E} \rightarrow \mathcal{F}$ such that $f(w) \subset v$ and $f(0) = 0$. We denote the category of relative exact categories and relative exact functors by \mathbf{RelEx} .

(3) We write \mathcal{E}^w for the full subcategory of \mathcal{E} consisting of those object x such that the canonical morphism $0 \rightarrow x$ is in w . We consider the following axioms.

(Strict axiom). \mathcal{E}^w is an exact category such that the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ is exact and reflects exactness.

(Very strict axiom). \mathbf{E} satisfies the strict axiom and the inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ induces a fully faithful functor $\mathcal{D}^b(\mathcal{E}^w) \hookrightarrow \mathcal{D}(\mathcal{E})$ on the bounded derived categories.

We denote the category of strict (resp. very strict) relative exact categories by $\mathbf{RelEx}_{\text{strict}}$ (resp. $\mathbf{RelEx}_{\text{vs}}$).

(4) A *relative natural equivalence* $\theta : f \rightarrow f'$ between relative exact functors $f, f' : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{E}' = (\mathcal{E}', w')$ is a natural transformation $\theta : f \rightarrow f'$ such that $\theta(x)$ is in w' for any object x in \mathcal{E} . Relative exact functors $f, f' : \mathbf{E} \rightarrow \mathbf{E}'$ are *weakly homotopic* if there is a zig-zag sequence of relative natural equivalences connecting f to f' . A relative functor $f : \mathbf{E} \rightarrow \mathbf{E}'$ is a *homotopy equivalence* if there is a relative exact functor $g : \mathbf{E}' \rightarrow \mathbf{E}$ such that gf and fg are weakly homotopic to identity functors respectively.

(5) A functor F from a full subcategory \mathcal{R} of \mathbf{RelEx} to a category \mathcal{C} is *categorical homotopy invariant* if for any relative exact functors $f, f' : \mathbf{E} \rightarrow \mathbf{E}'$ such that f and f' are weakly homotopic, we have the equality $F(f) = F(f')$.

Proposition 1.2. *Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be an exact functor between exact categories. If the induced functor $\mathcal{D}^\#(f) : \mathcal{D}^\#(\mathcal{F}) \rightarrow \mathcal{D}^\#(\mathcal{G})$ is fully faithful for some $\# \in \{b, \pm, \text{nothing}\}$, then $\mathcal{D}^\#(f)$ is fully faithful for any $\# \in \{b, \pm, \text{nothing}\}$.*

Proof. Since we have the fully faithful embeddings $\mathcal{D}^b(\mathcal{F}) \rightarrow \mathcal{D}^\pm(\mathcal{F})$ and $\mathcal{D}^\pm(\mathcal{F}) \rightarrow \mathcal{D}(\mathcal{F})$ and the equality $\mathcal{D}^+(\mathcal{F}) = (\mathcal{D}^-(\mathcal{F}^{\text{op}}))^{\text{op}}$, we only need to check that if $\mathcal{D}^b(f)$ (resp. $\mathcal{D}^-(f)$) is fully faithful, then $\mathcal{D}^+(f)$ (resp. $\mathcal{D}(f)$) is also. For any objects x and y in $\mathbf{Ch}^-(\mathcal{F})$ (resp. $\mathbf{Ch}(\mathcal{F})$), there are sequences $\mathbf{a} = \{a_0 \rightarrowtail a_1 \rightarrowtail a_2 \rightarrowtail \dots\}$ and $\mathbf{b} = \{b_0 \rightarrowtail b_1 \rightarrowtail b_2 \rightarrowtail \dots\}$ of inflations in $\mathbf{Ch}^b \mathcal{F}$ (resp. $\mathbf{Ch}^-(\mathcal{F})$) such that $x \xrightarrow{\sim} \mathbf{a}$ and $y \xrightarrow{\sim} \mathbf{b}$ in $\mathcal{T}(\mathfrak{F} \mathbf{Ch}^-(\mathcal{F}), \text{qis})$ (resp. $\mathcal{T}(\mathfrak{F} \mathbf{Ch} \mathcal{F}, \text{qis})$). Since we have the fully faithful embeddings $\mathcal{D}^\# \mathcal{F} \hookrightarrow \mathcal{T}(\mathfrak{F} \mathbf{Ch}^\# \mathcal{F}, \text{qis}) \hookleftarrow \mathcal{T}(\mathfrak{F} \mathbf{Ch}^{\#'} \mathcal{F}, \text{qis})$ where $\# = -$ and $\#' = b$ (resp. $\# = \text{nothing}$ and $\#' = +$) by [Sch06] Proposition 1 and Theorem 3, we regard both x and y as an objects in $\mathcal{T}(\mathfrak{F} \mathbf{Ch}^b \mathcal{F}, \text{qis})$ (resp. $\mathcal{T}(\mathfrak{F} \mathbf{Ch}^+ \mathcal{F}, \text{qis})$) and we have the natural isomorphism $\mathrm{Hom}_{\mathcal{D}^\# \mathcal{F}}(x, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}(\mathfrak{F} \mathbf{Ch}^{\#'} \mathcal{F}, \text{qis})}(x, y)$ where $\# = -$ and $\#' = b$ (resp. $\# = \text{nothing}$ and $\#' = +$). On the

other hand, the induced functor $\mathcal{T}(\mathfrak{F} \mathbf{Ch}^\#(f), \text{qis}) : \mathcal{T}(\mathfrak{F} \mathbf{Ch}^\#(\mathcal{F}), \text{qis}) \rightarrow \mathcal{T}(\mathfrak{F} \mathbf{Ch}^\#(\mathcal{G}), \text{qis})$ where $\# = b$ (resp. $\# = +$) is fully faithful by [Sch06] Corollary 2 and Proposition 1. Hence we obtain the result. \square

1.3 (Derived category). We define the *derived categories* of a strict relative exact category $\mathbf{E} = (\mathcal{E}, w)$ by the following formula

$$\mathcal{D}^\#(\mathbf{E}) := \text{Coker}(\mathcal{D}^\#(\mathcal{E}^w) \rightarrow \mathcal{D}^\#(\mathcal{E}))$$

where $\# = b, \pm$ or nothing. Namely $\mathcal{D}^\#(\mathbf{E})$ is a Verdier quotient of $\mathcal{D}^\#(\mathcal{E})$ by the thick subcategory of $\mathcal{D}^\#(\mathcal{E})$ spanned by the complexes in $\mathbf{Ch}^\#(\mathcal{E}^w)$.

Definition 1.4 (Exact sequence). A sequence $\mathbf{E} \xrightarrow{u} \mathbf{F} \xrightarrow{v} \mathbf{G}$ of strict relative exact categories is *exact* if the induced sequence of triangulated categories $\mathcal{D}^b(\mathbf{E}) \xrightarrow{\mathcal{D}^b(u)} \mathcal{D}^b(\mathbf{F}) \xrightarrow{\mathcal{D}^b(v)} \mathcal{D}^b(\mathbf{G})$ is exact. We sometimes denote the sequence above by (u, v) . For a full subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$, we let $E(\mathcal{R})$ denote the category of exact sequences in \mathcal{R} . We define three functors $s^{\mathcal{R}}, m^{\mathcal{R}}$ and $q^{\mathcal{R}}$ from $E(\mathcal{R})$ to \mathcal{R} which sends an exact sequence $\mathbf{E} \rightarrow \mathbf{F} \rightarrow \mathbf{G}$ to \mathbf{E} , \mathbf{F} and \mathbf{G} respectively

1.5 (Quasi-weak equivalences). Let $P^\# : \mathbf{Ch}^\#(\mathcal{E}) \rightarrow \mathcal{D}^\#(\mathbf{E})$ be the canonical quotient functor. We denote the pull-back of the class of all isomorphisms in $\mathcal{D}^\#(\mathbf{E})$ by $qw^\#$ or simply qw . We call a morphism in qw a *quasi-weak equivalence*. We write $\mathbf{Ch}^\#(\mathbf{E})$ for a pair $(\mathbf{Ch}^\#(\mathcal{E}), qw)$. We can easily prove that $\mathbf{Ch}^\#(\mathbf{E})$ is a complicial biWaldhausen category in the sense of [TT90, 1.2.11]. In particular, it is a relative exact category. The functor $P^\#$ induces an equivalence of triangulated categories $\mathcal{T}(\mathbf{Ch}^\#(\mathcal{E}), qw) \xrightarrow{\sim} \mathcal{D}^\#(\mathbf{E})$ (See [Sch11, 3.2.17]). If w is the class of all isomorphisms in \mathcal{E} , then qw is just the class of all quasi-isomorphisms in $\mathbf{Ch}^\#(\mathcal{E})$ and we denote it by qis .

Corollary 1.6. Let $\mathbf{E} = (\mathcal{E}, w)$ be a very strict relative exact category and $\# \in \{b, \pm, \text{nothing}\}$. Then

- (1) The inclusion functor $\mathcal{E}^w \hookrightarrow \mathcal{E}$ induces a fully faithful embedding $\mathcal{D}^\# \mathcal{E}^w \hookrightarrow \mathcal{D}^\# \mathcal{E}$.
- (2) The inclusion functor $\mathbf{Ch}^\# \mathcal{E}^w \hookrightarrow \mathbf{Ch}^\# \mathcal{E}$ and the identity functor on $\mathbf{Ch}^\# \mathcal{E}$ induce an exact sequence of relative exact categories $(\mathbf{Ch}^\# \mathcal{E}^w, \text{qis}) \rightarrow (\mathbf{Ch}^\# \mathcal{E}, \text{qis}) \rightarrow \mathbf{Ch}^\# \mathbf{E}$. \square

1.7 (Consistent axiom). Let $\mathbf{E} = (\mathcal{E}, w)$ be a strict relative exact category. There exists the canonical functor $\iota_\mathcal{E}^\# : \mathcal{E} \rightarrow \mathbf{Ch}^\#(\mathcal{E})$ where $\iota_\mathcal{E}^\#(x)^k$ is x if $k = 0$ and 0 if $k \neq 0$. We say that w (or \mathbf{E}) satisfies the *consistent axiom* if $\iota_\mathcal{E}^b(w) \subset qw$. We denote the full subcategory of consistent relative exact categories in \mathbf{RelEx} by $\mathbf{RelEx}_{\text{consist}}$.

Examples 1.8. (cf. [Moc13]). (1) A pair $(\mathcal{E}, i_\mathcal{E})$ of an exact category \mathcal{E} with the class of all isomorphisms $i_\mathcal{E}$ is a very strict consistent relative exact category.

(2) In particular we denote the trivial exact category by 0 and we also write $(0, i_0)$ for 0 . 0 is the zero objects in the category of consistent relative exact categories.

(3) A complicial exact category with weak equivalences in the sense of [Sch11, 3.2.9] is a consistent relative exact category. In particular for any relative exact category \mathbf{E} , $\mathbf{Ch}^\#(\mathbf{E})$ is a very strict consistent relative exact category.

Theorem 1.9 (Derived Gillet-Waldhausen theorem). (cf. [Moc13, 4.15]). Let \mathbf{E} be a consistent relative exact category. Then

- (1) The canonical functor $\iota_{\mathbf{Ch}^\#(\mathcal{E})} : \mathbf{Ch}^\#(\mathcal{E}) \rightarrow \mathbf{Ch}^b \mathbf{Ch}^\#(\mathcal{E})$ induces an equivalence of triangulated categories $\mathcal{D}^\#(\mathbf{E}) \xrightarrow{\sim} \mathcal{T}(\mathbf{Ch}^\#(\mathcal{E}), qw) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{Ch}^\#(\mathbf{E}))$.
- (2) In particular, the canonical functor $\iota_\mathcal{E}^b : \mathcal{E} \rightarrow \mathbf{Ch}^b(\mathcal{E})$ induces an equivalence of triangulated categories $\mathcal{D}^b(\mathbf{E}) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{Ch}^b(\mathbf{E}))$. \square

Definition 1.10 (Quotient of consistent relative exact categories). For any fully faithful relative functor $f : \mathbf{E} = (\mathcal{E}, w) \rightarrow \mathbf{F} = (\mathcal{F}, v)$ between consistent relative exact categories such that induced functor $\mathcal{D}^b(f)$ is also fully faithful, we define a *quotient* $\mathbf{F} / \mathbf{E} := (\mathbf{Ch}^b(\mathcal{F}), v/w)$ of \mathbf{F} by \mathbf{E} (along f) as follows. There exists a canonical quotient morphism $P_{\mathbf{F}/\mathbf{E}} : \mathbf{Ch}^b(\mathcal{F}) \rightarrow \mathcal{D}^b(\mathcal{F}) / \mathcal{D}^b(\mathcal{E})$. We write v/w for the pull back of all isomorphisms in $\mathcal{D}^b(\mathcal{F}) / \mathcal{D}^b(\mathcal{E})$ by $P_{\mathbf{F}/\mathbf{E}}$. We put $\mathbf{F} / \mathbf{E} := (\mathbf{Ch}^b(\mathcal{F}), v/w)$. \mathbf{F} / \mathbf{E} is again a consistent relative exact category by [Moc13, §4]. We have the canonical relative functor $\iota_{\mathbf{F}/\mathbf{E}}^b : \mathbf{F} \rightarrow \mathbf{F} / \mathbf{E}$.

2 Localizing theory

In this section, we will prove Theorem 0.2.

2.1. Let $\mathbf{E} = (\mathcal{E}, w)$ be a relative exact category. We denote the exact category of admissible short exact sequences in \mathcal{E} by $E(\mathcal{E})$. There exist three exact functors s, t and q from $E(\mathcal{E}) \rightarrow \mathcal{E}$ which send an admissible exact sequence $x \rightarrow y \rightarrow z$ to x, y and z respectively. We write $w_{E(\mathbf{E})}$ for the class of morphisms $s^{-1}(w) \cap t^{-1}(w) \cap q^{-1}(w)$ and put $E(\mathbf{E}) := (E(\mathcal{E}), w_{E(\mathbf{E})})$. We can easily prove that $E(\mathbf{E})$ is a relative exact category and the functors s, t and q are relative exact functors from $E(\mathbf{E})$ to \mathbf{E} . Moreover we can easily prove that if \mathbf{E} is consistent, then $E(\mathbf{E})$ is also consistent.

Now we give a definition of additive theories which is slightly different from [Moc13, 6.9, 7.8].

Definition 2.2 (Additive theory). (1) A full subcategory \mathcal{R} of \mathbf{RelEx} is *closed under extensions* if \mathcal{R} contains the trivial relative exact category 0 and if for any \mathbf{E} in \mathcal{R} , $E(\mathbf{E})$ is also in \mathcal{R} .
(2) Let F be a functor from a full subcategory \mathcal{R} of \mathbf{RelEx} closed under extensions to an additive category \mathcal{B} . We say that F is an *additive theory* if for any relative exact category \mathbf{E} in \mathcal{R} , the following projection is an isomorphism

$$\begin{pmatrix} F(s) \\ F(q) \end{pmatrix} : F(E(\mathbf{E})) \rightarrow F(\mathbf{E}) \oplus F(\mathbf{E}).$$

Lemma 2.3 (Eilenberg Swindle). Let \mathcal{R} be a full subcategory of $\mathbf{RelEx}_{\text{strict}}$ closed under extensions, F a categorical homotopy invariant additive theory from \mathcal{R} to an additive category \mathcal{B} and \mathbf{E} a strict relative exact category in \mathcal{R} . We assume that $\mathbf{Ch}^+ \mathbf{E}$ (resp. $\mathbf{Ch}^- \mathbf{E}$) is also in \mathcal{R} . Then $F(\mathbf{Ch}^+ \mathbf{E})$ (resp. $F(\mathbf{Ch}^- \mathbf{E})$) is trivial.

Proof. We only give a proof for $\mathbf{Ch}^+ \mathbf{E}$. We denote $f : \mathbf{Ch}^+ \mathbf{E} \rightarrow \mathbf{Ch}^+ \mathbf{E}$ to be a relative exact functor by sending an object x to $\bigoplus_{n \geq 0} x[2n]$. Then we have the equality $F(f[2]) + F(\text{id}_{\mathbf{Ch}^+ \mathbf{E}}) = F(f)$ and $F(f[2]) = F(f)$ by [Moc13] Proposition 7.9. Hence we obtain the result. \square

Definition 2.4 (Localization theory). A *localizing theory* (F, ∂) from a full subcategory \mathcal{R} of $\mathbf{RelEx}_{\text{strict}}$ to a triangulated category (\mathcal{T}, Σ) is a pair of functor $F : \mathcal{R} \rightarrow \mathcal{T}$ and a natural transformation $\partial : Fq \rightarrow \Sigma Fs$ between functors $E(\mathcal{R}) \xrightarrow{s} \mathcal{R} \xrightarrow{F} \mathcal{T}$ which sends a exact sequece $\mathbf{E} \xrightarrow{i} \mathbf{F} \xrightarrow{j} \mathbf{G}$ in \mathcal{R} to a distinguished triangle $F(\mathbf{E}) \xrightarrow{F(i)} F(\mathbf{F}) \xrightarrow{F(j)} F(\mathbf{G}) \xrightarrow{\partial(i,j)} \Sigma F(\mathbf{E})$ in \mathcal{T} .

Remark 2.5. (1) The non-connective K -theory on $\mathbf{RelEx}_{\text{consist}}$ studied in [Moc13] is a categorical homotopy invariant localization theory.

(2) (cf. [Moc13, 7.9]). Let F be a localization theory on a full subcategory \mathcal{R} . Then

- (i) F is a derived invariant functor. Namely if a morphism $\mathbf{E} \rightarrow \mathbf{F}$ in \mathcal{R} induces an equivalence of triangulated categories $\mathcal{D}^b \mathbf{E} \rightarrow \mathcal{D}^b \mathbf{F}$, then the induced morphism $F(\mathbf{E}) \rightarrow F(\mathbf{F})$ is an isomorphism. In particular if $\iota_{\mathcal{E}}^b : \mathbf{E} \rightarrow \mathbf{Ch}^b \mathbf{E}$ is in \mathcal{R} , then $F(\iota_{\mathcal{E}}^b)$ is an isomorphism.
- (ii) If further we assume that \mathcal{R} is closed under extensions and if F is categorical homotopy invariant, then we can easily prove that F is an additive theory.

Theorem 2.6. Let (F, ∂) be a categorical homotopy invariant localizing theory from a full subcategory \mathcal{R} closed under extensions to a triangulated category (\mathcal{T}, Σ) , \mathbf{E} a very strict relative exact category in \mathcal{R} . Assume that $\mathbf{Ch}^\# \mathbf{E}$ is also in \mathcal{R} for any $\# \in \{b, \pm, \text{nothing}\}$ and for any $\mathbf{F} \in \{\mathbf{E}, (\mathcal{E}, i_{\mathcal{E}}), (\mathcal{E}^w, i_{\mathcal{E}^w})\}$. Then there is an isomorphism $F \mathbf{Ch} \mathbf{E} \rightarrow \Sigma F \mathbf{Ch}^b \mathbf{E}$. In particular if further we assume that \mathbf{E} is consistent, then we have an isomorphism $F \mathbf{Ch} \mathbf{E} \xrightarrow{\sim} \Sigma F \mathbf{E}$.

Proof. First assume that w is the class of all isomorphisms in \mathcal{E} . Then the fully faithfull embeddings $\mathcal{D}^b(\mathcal{E}) \hookrightarrow \mathcal{D}^\pm(\mathcal{E})$ and $\mathcal{D}^\pm(\mathcal{E}) \hookrightarrow \mathcal{D}(\mathcal{E})$ yield the commutative diagram of distinguished triangles

$$\begin{array}{ccccccc} F(\mathbf{Ch}^b \mathcal{E}, \text{qis}) & \longrightarrow & F(\mathbf{Ch}^+ \mathcal{E}, \text{qis}) & \longrightarrow & F((\mathbf{Ch}^+ \mathcal{E}, \text{qis})/(\mathbf{Ch}^b \mathcal{E}, \text{qis})) & \xrightarrow{I} & \Sigma F(\mathbf{Ch}^b \mathcal{E}, \text{qis}) \\ \downarrow & & \downarrow & & \downarrow \text{III} & & \downarrow \\ F(\mathbf{Ch}^- \mathcal{E}, \text{qis}) & \longrightarrow & F(\mathbf{Ch} \mathcal{E}, \text{qis}) & \xrightarrow{\text{II}} & F((\mathbf{Ch} \mathcal{E}, \text{qis})/(\mathbf{Ch}^- \mathcal{E}, \text{qis})) & \longrightarrow & \Sigma F(\mathbf{Ch}^- \mathcal{E}, \text{qis}). \end{array}$$

Here the morphisms **I** and **II** are isomorphisms by triviality of $F(\mathbf{Ch}^\pm \mathcal{E}, \text{qis})$ and the morphism **III** is also an isomorphism by [Moc13] Proposition 7.10 (2). We denote the compositions of the morphisms **I** and the inverse of **III** and **II** by $\Delta_{\mathcal{E}} : F(\mathbf{Ch} \mathcal{E}, \text{qis}) \rightarrow \Sigma F(\mathbf{Ch}^b \mathcal{E}, \text{qis})$. Then $\Delta_{\mathcal{E}}$ is functorial on \mathcal{E} .

Next we consider the general case. By virtue of Corollary 1.6 (2), there is a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
F(\mathbf{Ch} \mathcal{E}^w, \text{qis}) & \longrightarrow & F(\mathbf{Ch} \mathcal{E}, \text{qis}) & \longrightarrow & F(\mathbf{Ch} \mathbf{E}) & \longrightarrow & \Sigma F(\mathbf{Ch} \mathcal{E}^w, \text{qis}) \\
\Delta_{\mathcal{E}^w} \downarrow \wr & & \wr \downarrow \Delta_{\mathcal{E}} & & \downarrow \text{IV} & & \wr \downarrow \Sigma \Delta_{\mathcal{E}^w} \\
\Sigma F(\mathbf{Ch}^b \mathcal{E}^w, \text{qis}) & \longrightarrow & \Sigma F(\mathbf{Ch}^b \mathcal{E}, \text{qis}) & \longrightarrow & \Sigma F(\mathbf{Ch}^b \mathbf{E}) & \longrightarrow & \Sigma^2 F(\mathbf{Ch}^b \mathcal{E}^w, \text{qis}).
\end{array}$$

Then there exists the dotted morphism **IV** in the diagram above which makes the diagram commutative and it is an isomorphism by 5-lemma. \square

Remark 2.7. The full subcategory $\mathbf{RelEx}_{\text{consist}}$ satisfies the assumption in Theorem 2.6 by [Moc13]. In particular we obtain Theorem 0.2.

3 Higher derived categories

In this section, we assume that \mathbf{E} is a very strict consistent relative exact category.

3.1 (Higher derived categories). Let us denote n -th times iteration of \mathbf{Ch} for \mathbf{E} by $\Sigma^n \mathbf{E}$ and $\mathcal{D}_n(\mathbf{E}) := \mathcal{D}^b(\Sigma^n \mathbf{E})$ the n -th higher derived category of \mathbf{E} . Then

- (1) $\mathcal{D}_0(\mathbf{E})$ is just the usual bounded derived category $\mathcal{D}^b(\mathbf{E})$ of \mathbf{E} .
- (2) For any positive integer n , $\mathcal{D}_n(\mathbf{E})$ is the unbounded derived category $\mathcal{D}(\Sigma^{n-1} \mathbf{E})$ of $\Sigma^{n-1} \mathbf{E}$ by Theorem 1.9 (1). In particular, $\mathcal{D}_1(\mathbf{E})$ is just the unbounded derived category $\mathcal{D}(\mathbf{E})$ of \mathbf{E} .

As the following corollary, we can consider the negative K -groups as an obstruction group of idempotent completeness of the higher derived categories:

Corollary 3.2. *For any positive integer n , we have*

- (1) $\mathbb{K}_{-n}(\mathbf{E}) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathbf{E}))$.
- (2) $\mathbb{K}_{-n}(\mathbf{E})$ is trivial if and only if $\mathcal{D}_n(\mathbf{E})$ is idempotent complete.

Proof. By Theorem 0.2, we have $\mathbb{K}_{-n}(\mathbf{E}) \simeq \mathbb{K}_0(\mathbf{Ch}_n(\mathbf{E})) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathbf{E}))$. Then Proposition 3.3 below leads the desired assertion. \square

Proposition 3.3. (1) *For an essentially small triangulated category \mathcal{T} , if $\mathbb{K}_0(\mathcal{T}) = K_0(\mathcal{T}^\sim)$ is trivial, then \mathcal{T} is idempotent complete.*

(2) *The derived category $\mathcal{D}(\mathbf{E})$ is idempotent complete if and only if the Grothendieck group $\mathbb{K}_0(\mathcal{D}(\mathbf{E})) = K_0(\mathcal{D}(\mathbf{E})^\sim)$ is trivial.*

Proof. (1) Since the map $K_0(\mathcal{T}) \rightarrow K_0(\mathcal{T}^\sim)$ is injective by [Tho97] Corollary 2.3, now $K_0(\mathcal{T})$ is also trivial. Applying the Thomason classification theorem of (strictly) dense triangulated subcategories in essentially small triangulated categories [Tho97] Theorem 2.1 for \mathcal{T}^\sim , the inclusion functor $\mathcal{T} \rightarrow \mathcal{T}^\sim$ must be an equivalence.

(2) We have the equalities $K_0(\mathbf{Ch}(\mathcal{E}^w), \text{qis}) = K_0(\mathbf{Ch}(\mathcal{E}), \text{qis}) = 0$ as in the proof of Corollary 6 in [Sch06]. Therefore $K_0(\mathcal{D}(\mathbf{E})) = K_0(\mathbf{Ch}(\mathbf{E})) = 0$ by the canonical fibration sequence associated to the exact sequence in Corollary 1.6 (2) for $\mathbf{Ch} \mathbf{E}$. If $\mathcal{D}(\mathbf{E})$ is idempotent complete, that is, $\mathcal{D}(\mathbf{E}) \xrightarrow{\sim} \mathcal{D}(\mathbf{E})^\sim$, then we have $\mathbb{K}_0(\mathcal{D}(\mathbf{E})) = K_0(\mathcal{D}(\mathbf{E})^\sim) = K_0(\mathcal{D}(\mathbf{E})) = 0$. The converse is followed from (1). \square

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